QUATERNION ROTATION SUMMARY

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1. Introduction

In 1843 William Hamilton invented the hypercomplex number of rank 4 called the quaternion. The rule he developed

$$i^2 = j^2 = k^2 = ijk = -1$$

was critical for its invention. Quaternions consist of a scalar part and a vector part, and hence exist in $\mathbb{R}^4$.

$$\mathring{q} = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k}$$

A subset of this space are the quaternions that have 0 as their scalar part (essentially a vector only) and they are called pure quaternions and can be mapped directly to vectors in $\mathbb{R}^3$. The rules that govern the bijective transformation between pure quaternions and vectors are covered in Section 2.1

1.1. Quaternion Multiplication.

Multiplication of quaternions must obey the rules

$$\begin{aligned} &i^2 = j^2 = k^2 = ijk = -1 \\ &ij = k \quad ji = -k \\ &ki = j \quad ik = -j \\ &jk = i \quad kj = -i \end{aligned}$$

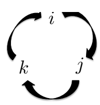

Figure 1. Multiplication circle, clockwise is positive, counter is negative, all three is negative

So for quaternions $\mathring{p} = p_0 + \mathbf{p}$ and $\mathring{q} = q_0 + \mathbf{q}$, multiplication is:

$$\begin{aligned} \mathring{r} = \mathring{p}\mathring{q} &= (p_0 + p_1\hat{i} + p_2\hat{j} + p_3\hat{k})(q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k}) \\ &= p_0q_0 + \hat{i}p_0q_1 + \hat{j}p_0q_2 + \hat{k}p_0q_3 \\ &+ \hat{i}p_1q_0 + \hat{i}^2p_1q_1 + \hat{i}\hat{j}p_1q_2 + \hat{i}\hat{k}p_1q_3 \\ &+ \hat{j}p_2q_0 + \hat{j}\hat{i}p_2q_1 + \hat{j}^2p_2q_2 + \hat{j}\hat{k}p_2q_3 \\ &+ \hat{k}p_3q_0 + \hat{k}\hat{i}p_3q_1 + \hat{k}\hat{j}p_3q_2 + \hat{k}^2p_3q_3 \\ &= (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3) + (p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2)\hat{i} \\ &+ (p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1)\hat{j} + (p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0)\hat{k} \\ &= p_0q_0 - \mathbf{p}\cdot\mathbf{q} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q} \end{aligned}$$

or in matrix form:

$$\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

1.2. **Quaternion Complex Conjugate, Norm and Inverse.**

Since i, j, k are imaginary numbers, the complex conjugate is the same as the tradition i(or j):

$$\mathring{q} = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k}$$
$$\mathring{q}^* = q_0 - q_1\hat{i} - q_2\hat{j} - q_3\hat{k}$$

The norm of the quaternion is essentially the length of the quaterion [2]:

$$N(\mathring{q}) = \sqrt{\mathring{q}^*\mathring{q}} = \sqrt{\mathring{q}\mathring{q}^*}$$

Shown:

$$\begin{aligned} N^2(\mathring{q}) &= (q_0 - \mathbf{q})(q_0 + \mathbf{q}) \\ &= q_0q_0 - (-\mathbf{q})\cdot\mathbf{q} + q_0\mathbf{q} + (-\mathbf{q})q_0 + (-\mathbf{q})\times\mathbf{q} \\ &= q_0^2 + \mathbf{q}\cdot\mathbf{q} \\ &= q_0^2 + q_1^2 + q_2^2 + q_3^2 \\ &= |\mathring{q}|^2 \end{aligned}$$

and for a product of two the norm is the multiplication of both individual norms [2]:

$$\begin{aligned} N^2(\mathring{p}\mathring{q}) &= (\mathring{p}\mathring{q})(\mathring{p}\mathring{q})^* \\ &= \mathring{p}\mathring{q}\mathring{q}^*\mathring{p}^* \\ &= \mathring{p}N^2(\mathring{q})\mathring{p}^* \\ &= \mathring{p}\mathring{p}^*N^2(\mathring{q}) \ \textit{(as the norm is a scalar)} \\ &= N^2(\mathring{p})N^2(\mathring{q}) \end{aligned}$$

Now that we have defined the norm we can now investigate the inverse. We want the inverse to be such that

$$\mathring{q}^{-1}\mathring{q} = \mathring{q}\mathring{q}^{-1} = 1$$

by pre or post multiplying by $\mathring{q}^*$ [2]:

$$\mathring{q}^{-1}\mathring{q}\mathring{q}^* \equiv \mathring{q}^*\mathring{q}\mathring{q}^{-1} = \mathring{q}^* \ \textit{and since } \mathring{q}\mathring{q}^* = N^2(\mathring{q})$$
$$\mathring{q}^{-1} = \frac{\mathring{q}^*}{N^2(\mathring{q})} = \frac{\mathring{q}^*}{|\mathring{q}|^2}$$

2. Rotation and Transformations

2.1. **Quaternion Rotation.**

As mentioned before quaternions exist in $\mathbb{R}^4$. When a quaternion is multiplied by a vector then essentially the vector is a quaternion with scalar 0, and the result is not garunteed to be in $\mathbb{R}^3$. If 2 quaternions q and r were multplied by a vector (quat with scalar 0: pure quaternion) p, then the possible combinations would be [2]:

$$\begin{matrix} p\mathring{q}\mathring{r} & \mathring{q}\mathring{r}p & \mathring{r}p\mathring{q} \\ p\mathring{r}\mathring{q} & \mathring{r}\mathring{q}p & \mathring{q}p\mathring{r} \end{matrix}$$

The products of either $\mathring{q}\mathring{r}$ or $\mathring{r}\mathring{q}$ would again be a quaternion and multiplication by p would be closed under $\mathbb{R}^4$ but not under set of pure quaternions (you wouldn't be garunteed a pure quaternion). So we are left with the triples qpr or rpq. Expanding this multplication we see that for $\mathring{q} = q_0 + \mathbf{q}$, $\mathring{p} = 0 + \mathbf{p}$, $\mathring{r} = r_0 + \mathbf{r}$, the real part of qpr is:

$$-r_0(\mathbf{q} \cdot \mathbf{p}) - q_0(\mathbf{p} \cdot \mathbf{r}) - (\mathbf{q} \times \mathbf{p}) \cdot \mathbf{r}$$

and using rules of vector algebra this scalar portion may be expressed as [2]:

$$-r_0(\mathbf{q} \cdot \mathbf{p}) - q_0(\mathbf{r} \cdot \mathbf{p}) + (\mathbf{q} \times \mathbf{r}) \cdot \mathbf{p}$$

if we want the output to be a pure quaternion, then this real part must be zero, which is true if $\mathbf{r} = -\mathbf{q}$ meaning that [2]:

$$\mathring{r} = r_0 + \mathbf{r} = q_0 - \mathbf{q} = \mathring{q}^* \implies \mathring{q} = \mathring{r}^*$$

Hence the multiplication

$$\mathbf{w_1} = \mathring{q}\mathbf{v}\mathring{q}^*$$
$$\mathbf{w_2} = \mathring{q}^*\mathbf{v}\mathring{q}$$

is closed under pure quaternions. And our only task remaining is to see if we can bridge such an action on a vector to a rotation of the vector.

During the pure rotation of a vector, the length of the vector is maintained, the above multiplication is only garunteed to maintain vector length if the quaternion $\mathring{q}$ has a norm of 1. So we know we need:

$$|\mathring{q}| = q_0^2 + |\mathbf{q}|^2 = 1$$

Realizing that for any angle θ we have the trigonometric relationship:

$$cos(\theta)^2 + sin(\theta)^2 = 1$$

Then we can equate:

$$cos^2(\theta) = q_0$$
$$sin^2(\theta) = |\mathbf{q}|^2$$

The above assertion is critical to rotation. Now suppose there is some vector $\mathbf{u}$ (which will be the axis of rotatation) that is the normalized vector portion of the quaternion [2]:

$$u = \frac{\mathbf{q}}{|\mathbf{q}|} = \frac{\mathbf{q}}{sin(\theta)}$$

Then the unit quaternion can be written as:

$$\mathring{q} = q_0 + \mathbf{q} = cos(\theta) + \mathbf{u}sin(\theta)$$

(and note that rotating in the other direction $-\theta$, will be the conjugate of the quaternion)[2]:

$$cos(-\theta) + \mathbf{u}sin(-\theta) = cos(\theta) - \mathbf{u}sin(\theta) = \mathring{q}^*$$

note also that multiplying two rotational quaternions will produce a third quaternion which is a combination of the two rotations. Below let quaternions $\mathring{p}$ and $\mathring{p}$ both share their axis of rotation $\mathbf{u}$ [2]:

$$\begin{aligned}
\mathring{p} &= cos(\alpha) + \mathbf{u}sin(\alpha) \\
\mathring{q} &= cos(\beta) + \mathbf{u}sin(\beta) \\
\mathring{r} = \mathring{p}\mathring{q} &= (cos(\alpha) + \mathbf{u}sin(\alpha))(cos(\beta) + \mathbf{u}sin(\beta)) \\
&= cos(\alpha)cos(\beta) - (\mathbf{u}sin(\alpha)) \cdot (\mathbf{u}sin(\beta)) \\
&+ cos(\alpha)(\mathbf{u}sin(\beta)) + cos(\beta)(\mathbf{u}sin(\alpha)) \\
&+ \mathbf{u}sin(\alpha) \times \mathbf{u}sin(\beta) \\
&= cos(\alpha)cos(\beta) - sin(\alpha)sin(\beta) + \mathbf{u}(sin(\alpha)cos(\beta) + cos(\alpha)sin(\beta)) \\
&= cos(\alpha+\beta) + \mathbf{u}sin(\alpha+\beta) \\
&= cos(\gamma) + \mathbf{u}sin(\gamma) = \mathring{r}
\end{aligned}$$

The last order of business to to mind the angle of rotation in the rotational quaternion. Take for instance the example presented in [2], if we wanted to rotate the vector $\mathbf{v} = 1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$ by angle $\theta = \frac{\pi}{6}$ for quaternion $\mathring{q} = cos(\theta) + \mathbf{k}sin(\theta) = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathbf{k}$, then we have:

$$\begin{aligned}
\mathring{w} &= \mathring{q}\mathbf{v}\mathring{q}^* \\
&= (\frac{\sqrt{3}}{2} + \frac{1}{2}\mathbf{k})(0 + \mathbf{i})(\frac{\sqrt{3}}{2} - \frac{1}{2}\mathbf{k}) \\
&= (\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j})(\frac{\sqrt{3}}{2} - \frac{1}{2}\mathbf{k}) \\
&= \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}
\end{aligned}$$

So we see that we again obtained a pure quaternion (vector), but notice that the angle of rotation is $\theta = \frac{\pi}{3}$ (as $cos\frac{\pi}{3} = \frac{1}{2}$). Now notice that w is a unit vector, but the angle between this w and v is $\frac{\pi}{3}$ which is double the desired $\frac{\pi}{6}$ so essentially what we had was:

$$w = \mathbf{i}cos(2\theta) + \mathbf{j}sin(2\theta)$$

For this reason when representing the rotation quaternion we will divide the angle by two in order to achieve the desired rotation angle we desire:

$$\boxed{\mathring{q} = cos(\frac{\theta}{2}) + \mathbf{u}sin(\frac{\theta}{2})}$$

Note the rotation form we have when rotating vector $\mathbf{v}$ into $\mathbf{w}$:

$$\begin{aligned}
\mathbf{w} = \mathring{q}\mathbf{v}\mathring{q}^* &= (q_0 + \mathbf{q})(0 + \mathbf{v})(q_0 - \mathbf{q}) \\
&= (q_0^2 - |\mathbf{q}|^2)\mathbf{v} + 2(\mathbf{q} \cdot \mathbf{v})\mathbf{q} + 2q_0(\mathbf{q} \times \mathbf{v})
\end{aligned}$$

note that the axis of rotation is invariant (if the vector $\mathbf{v} = k\mathbf{q}$ lies on the axis of rotation ($\mathbf{q}$), then it doesn't change), which shows $\mathbf{u}$ is the axis of rotation [2]:

$$\begin{aligned}
\mathbf{w} &= \mathring{q}\mathbf{v}\mathring{q}^* \\
&= \mathring{q}k\mathbf{q}\mathring{q}^* \\
&= (q_0^2 - 1)(k\mathbf{q}) + 2(\mathbf{q} \cdot k\mathbf{q})\mathbf{q} + 2q_0(\mathbf{q} \times k\mathbf{q}) \\
&= kq_0^2\mathbf{q} - k|\mathbf{q}|^2\mathbf{q} + 2k|\mathbf{q}|^2\mathbf{q} \\
&= k(q_0^2 + |\mathbf{q}|^2)\mathbf{q} \\
&= k\mathbf{q}
\end{aligned}$$

Quaternion rotations may also be represented as matricies [2] :

$$(q_0^2 - |\mathbf{q}|^2)\mathbf{v} = \begin{bmatrix} (q_0^2 - q_1^2 - q_2^2 - q_3^2) & 0 & 0 \\ 0 & (q_0^2 - q_1^2 - q_2^2 - q_3^2) & 0 \\ 0 & 0 & (q_0^2 - q_1^2 - q_2^2 - q_3^2) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$2(\mathbf{v}\cdot\mathbf{q})\mathbf{q} = \begin{bmatrix} 2q_1^2 & 2q_1q_2 & 2q_1q_3 \\ 2q_1q_2 & 2q_2^2 & 2q_2q_3 \\ 2q_1q_3 & 2q_2q_3 & 2q_3^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$2q_0(\mathbf{q}\times\mathbf{v}) = \begin{bmatrix} 0 & -2q_0q_3 & 2q_0q_2 \\ 2q_0q_3 & 0 & -2q_0q_1 \\ -2q_0q_2 & 2q_0q_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

hence

$$\mathbf{w} = \mathring{q}\mathbf{v}\mathring{q}^*$$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\mathbf{w} = \mathring{q}\mathbf{v}\mathring{q}^* = Q\mathbf{v}$$

$$\mathbf{w}' = \mathring{q}^*\mathbf{v}\mathring{q} = Q^t\mathbf{v} \textit{ is rotated in opp direction around } \mathring{q}.$$

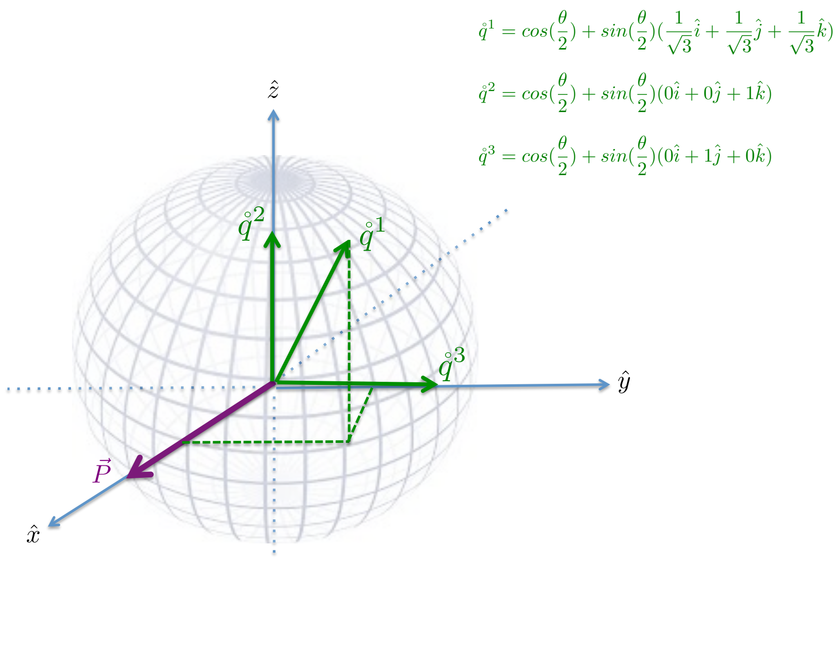

FIGURE 2. The sphere in the background is the unit sphere (radius is 1 in $\mathbb{R}^3$ which is norm of $\mathbf{u}$), note that the unit sphere in $\mathbb{R}^4$ contains unit quaternions (norm 1), hence rotational quaternions under multiplication stay to this surface (combining multiple rotations gives you another rotation) $\mathring{q} = cos(\theta) + sin(\theta)\mathbf{u}$. Given some vector like $\vec{P} = 1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$ if it were rotated by $\theta = \frac{2\pi}{3}$ about $\mathring{q}^1$ then it would land on y axis. If $\vec{P}$ were rotated about $\mathring{q}^2$ by angle $\theta = \frac{\pi}{2}$, then it would land on the y axis again. If $\vec{P}$ were rotated about $\mathring{q}^3$ through angle $\theta = -\frac{\pi}{2}$ then it would land on z axis.

2.2. Quaternion Transformation.

Transformations allow us to not only rotate vectors but translate them as well. Corke presents the quaternion based transformation variable ξ in [1]:

$$\xi(\vec{t}, \mathring{q})$$

Rules governing the transformation operation on vector $\vec{r}$ are [1]:

$$\xi(\vec{r}) = \mathring{q}\vec{r}\mathring{q}^* + \vec{t}$$

if we want to combine a series of transformations, the composition is [1]:

$$\xi_1\xi_2 = (\vec{t}_1 + \mathring{q}_1\vec{t}_2\mathring{q}_1^*, \mathring{q}_1\mathring{q}_2)$$

Transformation Example. The example below in Figure 3 shows the transformation from tip to tip of the vector $\vec{P}^1$ to $\vec{P}^2$ (note that if you wanted to think of the vector of length 1 spatially it would be equivalent to $\vec{P}^1 - \vec{0}$ (the origin (0,0)), and $\vec{P}^2 - \vec{t}$, but the tip gives you reference in world frame).

$$\vec{P}^1 = 1\hat{\imath} + 0\hat{\jmath} + 0\hat{k} \qquad \vec{u} = \frac{1}{\sqrt{3}}\hat{\imath} + \frac{1}{\sqrt{3}}\hat{\jmath} + \frac{1}{\sqrt{3}}\hat{k} \qquad \theta = \frac{2\pi}{3} \qquad \vec{t} = -1\hat{\imath} + 4\hat{\jmath} + 1\hat{k}$$

$$\vec{P}^2 = -1\hat{\imath} + 5\hat{\jmath} + 1\hat{k}$$

$$\mathring{q} = q_0 + q_1\hat{\imath} + q_2\hat{\jmath} + q_3\hat{k} \qquad \mathring{q} = cos(\frac{\theta}{2}) + sin(\frac{\theta}{2})\vec{u} \qquad \mathring{q} = \frac{1}{2} + \frac{1}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath} + \frac{1}{2}\hat{k}$$

Transformation: (arrow tip to arrow tip) $\quad \xi(\vec{t}, \mathring{q})$

$$\vec{P}^2 = \xi(\vec{P}^1) = \mathring{q}(\vec{P}^1) + \vec{t} = \mathring{q}\vec{P}^1\mathring{q}^* + \vec{t} = (\frac{1}{2} + \frac{1}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath} + \frac{1}{2}\hat{k})(1\hat{\imath} + 0\hat{\jmath} + 0\hat{k})(\frac{1}{2} - \frac{1}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath} - \frac{1}{2}\hat{k}) + -1\hat{\imath} + 4\hat{\jmath} + 1\hat{k}$$

$$= (\frac{1}{2} + \frac{1}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath} + \frac{1}{2}\hat{k})(\frac{1}{2} + \frac{1}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath} - \frac{1}{2}\hat{k}) + -1\hat{\imath} + 4\hat{\jmath} + 1\hat{k}$$

$$= \frac{1}{4}\hat{\imath} + \frac{1}{4} - \frac{1}{4}\hat{k} + \frac{1}{4}\hat{\jmath} + \frac{1}{4}\hat{\imath}^2 + \frac{1}{4}\hat{\imath} - \frac{1}{4}\hat{\imath}\hat{k} + \frac{1}{4}\hat{\imath}\hat{\jmath} + \frac{1}{4}\hat{\jmath} + \frac{1}{4}\hat{\jmath}\hat{\imath} - \frac{1}{4}\hat{\jmath}\hat{k} + \frac{1}{4}\hat{\jmath}^2 + \frac{1}{4}\hat{k} + \frac{1}{4}\hat{k}\hat{\imath} - \frac{1}{4}\hat{k}^2 + \frac{1}{4}\hat{k}\hat{\jmath} + -1\hat{\imath} + 4\hat{\jmath} + 1\hat{k}$$

$$= (\frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4}) + (\frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4})\hat{\imath} + (\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4})\hat{\jmath} + (-\frac{1}{4} + \frac{1}{4} - \frac{1}{4} + \frac{1}{4})\hat{k} + -1\hat{\imath} + 4\hat{\jmath} + 1\hat{k}$$

$$= -1\hat{\imath} + 5\hat{\jmath} + 1\hat{k} = \vec{P}^2$$

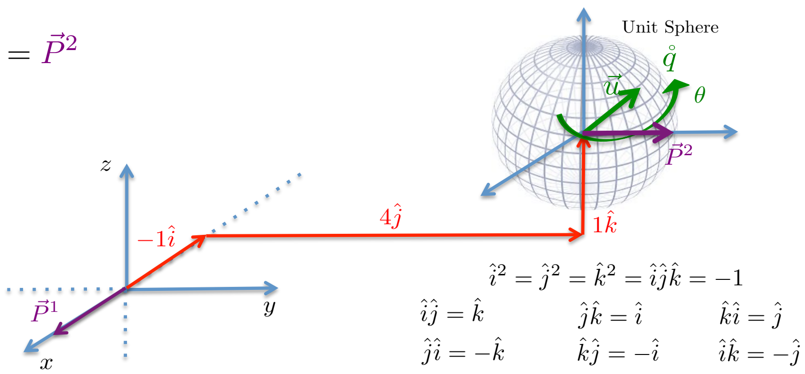

FIGURE 3. Quaternion Transformation Example: in a quaternion transformation of a vector, the vector is translated, then in the new frame a unit sphere (radius of 1) can be considered (as rotational quaternion has unit length), and quaternion vector portion is the axis of rotation with angle θ.

3. Further Reading

Reference [2] is especially helpful for understanding quaternion operations, and rotations, and reference [1] for transformations using quaternions as well as computer vision techniques in robotics.

References

[1] Peter Corke. *Robotics, Vision and Control Fundamental Algorithms in MATLAB*. Springer, 2011.
[2] Jack Kuipers. *Quaternions and Rotation Sequences: A Primer with applications to Orbits, Aerospace, and Virtual Reality*. Princeton University Press, 1999.